

# Lyapunov Exponent and Reissner Nordström Black Hole

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## Abstract

We explicitly derive the principal Lyapunov Exponent *in terms of the radial equation of ISCO*(Innermost Stable Circular Orbit) for Spherically symmetry(Schwarzschild, Reissner Nordström) black-hole space-times. Using it, we show that the ISCO occurs at  $r_{ISCO} = 4M$  for extremal Reissner Nordström black-hole and  $r_{ISCO} = 6M$  for Schwarzschild black-hole. We elucidate the connection between Lyapunov Exponent and *Geodesic Deviation Equation*. We also compute the *Kolmogorov-Sinai(KS)* entropy which measures the rate of exponential divergence between two trajectories(geodesics)via Lyapunov Exponent. We further prove that ISCO is characterized by the *greatest* possible orbital period i.e.  $T_{ISCO} > T_{photon}$  among all types of circular geodesics(both time-like and null, geodesic and non-geodesic) as measured by the asymptotic observers. Therefore, ISCO provide the *slowest way* to circle the black hole.

## 1 Introduction

Nonlinearity of Einsteins equation makes nonlinearity of Einsteins General Theory of Relativity. So there may be a certain link between nonlinear Einsteins General Theory of Relativity and nonlinear dynamics. Particularly Lyapunov Exponent and Kolmogorov-Sinai(KS) entropy, are one of the bridges between them. In this paper, we shall focus on analytical calculations involving Lyapunov Exponents and Kolmogorov-Sinai in terms of equations of circular geodesics around a black-hole spacetime. This equatorial circular geodesics around a black-hole space-time playing a crucial role in General Relativity for classification of the orbits. It also determines important feature of the space-times and gives important information on the back ground geometry.

The Lyapunov Exponent has been used to probe the instability of circular null geodesics and in terms of the Quasinormal modes for spherically symmetric space-time in[7], but the focus there is on null circular geodesics. It has been shown in this reference

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that in the eikonal approximation, the real and imaginary parts of the Quasi Normal Modes of any spherically symmetric, asymptotically flat spacetime are given by (multiples of) the frequency and instability time scale of the unstable circular null geodesics.

Note however that the Principal Lyapunov Exponents ( $\lambda$ ) have been computed in [5]-[7] using a *coordinate* time  $t$  where  $t$  is measured by asymptotic observers. Thus, these exponents are explicitly coordinate dependent, and therefore have a degree of unphysicality. Here we compute the Principal Lyapunov Exponent ( $\lambda$ ) and Kolmogorov-Sinai entropy  $h_{ks}$  analytically by using the *proper time*  $\tau$  which is coordinate invariant. Using them we study the stability of equatorial circular geodesics for Reissner Nordström and Schwarzschild black-hole space-time. Another interesting point we have studied here is that the Lyapunov Exponent and Kolmogorov-Sinai entropy  $h_{ks}$  can be expressed in terms of the ISCO equation and we find the ISCO occurs at  $r_{ISCO} = 4M$  for extremal Reissner Nordström (RN) black-hole and  $r_{ISCO} = 6M$  for Schwarzschild black-hole *via* Lyapunov stability analysis and Kolmogorov-Sinai entropy  $h_{ks}$ .

The paper is organized as follows. In section 2 we provide the fundamentals of the Lyapunov Exponent. In section 3, the Lyapunov Exponent has been expressed in terms of the standard effective potential. In section 4 we elucidate the connection between Lyapunov Exponent and Kolmogorov-Sinai Entropy which is also determine the exponential divergence between neighbouring trajectories. In section 5 we derive the standard effective potential in the equatorial plane. In section 6 we fully describe the equatorial circular geodesics, both time-like and null, for Reissner Nordström blackhole and also discuss the Lyapunov Exponent can be expressed in terms of the ISCO equation and studied for the stability of the circular geodesics. In section 7 we discuss similar things for extremal Reissner Nordström spacetime. In section 8 we relates the Geodesic Deviation equation in terms of Lyapunov Exponent for any spherically symmetric spacetime. In section 9 we compare the ratio of angular velocity between ISCO and null circular geodesics. In section 10, we present our conclusions.

## 2 Fundamentals of Lyapunov Exponent:

The Lyapunov Exponents in a classical phase space are a measure of the average rates of expansion and contraction of a trajectories surrounding it. They are asymptotic quantities defined locally in state space, and describe the exponential rate at which a perturbation to a trajectory of a system grows or decays with time at a certain location in the state space. A positive Lyapunov Exponent indicates a divergence between two nearby geodesics, the paths of such a system are extremely sensitive to changes of the initial conditions. A negative Lyapunov Exponent implies a convergence between two nearby geodesics and the vanishing Lyapunov Exponent indicates the existence of periodic motion i.e. the system has a limit cycle. Lyapunov Exponents can also distinguish among fixed points, periodic motions, quasi-periodic motions, and chaotic motions.

An  $n$ -dimensional continuous-time (autonomous) smooth dynamical system governs by the differential equation [15] of the form

$$\frac{dx}{dt} = F(x; M) . \quad (1)$$

The vector  $x$  consists  $n$  state variables, the function  $F$  describes the non-linear evolution of the dynamical system and  $M$  is a vector control parameter. The solutions are fixed point solutions or equilibrium solutions. Where  $t$  is defined as time parameter. The fixed points or critical points, are defined by the vanishing of the vector field; that is

$$F(x; M) = 0 . \quad (2)$$

Let the solution of (2) for  $M = M_0$  be  $x_0$ , where  $x_0 \in \mathcal{R}^n$  and  $M_0 \in \mathcal{R}^m$ . To calculate the stability of this equilibrium solutions, we simply apply on it a small perturbation  $y$  and obtain

$$x(t) = x_0 + y(t) . \quad (3)$$

Substituting (3) into (1) yields

$$\frac{dy}{dt} = F(x_0 + y; M) . \quad (4)$$

Note that the fixed point  $x = x_0$  of (1) has been transformed into the fixed point  $y = 0$  of (4). Expanding (4) in a Taylor series about  $x_0$  and keeping only linear terms in the perturbation leads to

$$\frac{dy}{dt} = F(x_0; M_0) + \frac{\partial F(x_0; M_0)}{\partial x} y + O(\|y\|^2) . \quad (5)$$

or

$$\frac{dy}{dt} = \frac{\partial F(x_0; M_0)}{\partial x} y = Ay . \quad (6)$$

where the matrix  $A$  is called Jacobian matrix. If the components of  $F$  are  $F_1(x_1, x_2, x_3, \dots, x_n)$ ,  $F_2(x_1, x_2, x_3, \dots, x_n)$ ,  $F_3(x_1, x_2, x_3, \dots, x_n)$  then

$$A = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \dots & \frac{\partial F_1}{\partial x_n} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & \dots & \frac{\partial F_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial x_1} & \frac{\partial F_n}{\partial x_2} & \dots & \frac{\partial F_n}{\partial x_n} \end{pmatrix} . \quad (7)$$

The eigen values of the constant matrix  $A$  provide information about the local stability of the fixed point  $x_0$ . The solution of the equation (6) becomes

$$y(t) = e^{(t-t_0)A}y_0 . \quad (8)$$

where

$$e^{(t-t_0)A}y_0 = \sum_{i=0}^{\infty} \frac{(t-t_0)^i}{i!} A^i . \quad (9)$$

and  $y_0 \in \mathcal{R}^n$  at time  $t_0 \in \mathcal{R}$ . If the eigen values  $\lambda_i$  of the matrix  $A$  are distinct, then there exists a matrix  $P$  such that  $P^{-1}AP = D$ , where  $D$  is a diagonal matrix with eigen values  $\lambda_1, \lambda_2, \dots, \lambda_n$ ; that is

$$D = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & \lambda_n \end{pmatrix} . \quad (10)$$

If the eigen values are complex, then the matrix  $P$  will also be complex. The columns of the matrix  $P$  are the right eigen vectors  $p_1, p_2, \dots, p_n$  of the matrix  $A$  corresponding to the eigen values  $\lambda_1, \lambda_2, \dots, \lambda_n$ ; that is,

$$P = \begin{pmatrix} p_1 \\ p_2 \\ \cdot \\ \cdot \\ \cdot \\ p_n \end{pmatrix} . \quad (11)$$

Hence

$$AP = \begin{pmatrix} Ap_1 \\ Ap_2 \\ \cdot \\ \cdot \\ \cdot \\ Ap_n \end{pmatrix} = \begin{pmatrix} \lambda_1 p_1 \\ \lambda_2 p_2 \\ \cdot \\ \cdot \\ \cdot \\ \lambda_n p_n \end{pmatrix} = PD . \quad (12)$$

Consequently

$$D = P^{-1}AP . \quad (13)$$

Now transforming  $y = Pz$  into (6), we get

$$\begin{aligned} P \frac{dz}{dt} &= APz \\ &\quad \text{or} \\ \frac{dz}{dt} &= Dz . \end{aligned} \quad (14)$$

Hence

$$z = e^{(t-t_0)D} z_0 . \quad (15)$$

where

$$z_0 = z(t_0) = P^{-1}y_0 . \quad (16)$$

In terms of  $y$ , this solution becomes

$$y(t) = P e^{(t-t_0)D} P^{-1} y_0 . \quad (17)$$

The matrix  $e^{(t-t_0)D}$  is a diagonal matrix entries  $e^{(t-t_0)\lambda_i}$ . Therefore the eigen values of  $A$  are also known as Characteristic Exponents or Lyapunov Exponents associated with  $F$  at  $(x_0, M_0)$ .

If we consider an initial deviation  $y(0)$ , its evolution is described by

$$y(t) = \Phi(t)y(0) . \quad (18)$$

where  $\Phi(t)$  is the fundamental(transition) matrix solution (6) associated with the trajectory say  $X(t)$  which governs the dynamical equation (1). For an appropriate chosen  $y(0)$  in (18) the rate of exponential expansion or contraction in the direction of  $y(0)$  on the trajectory passing through  $X_0$  (trajectory at  $t = 0$ ) is given by

$$\lambda_i^t = \lim_{t \rightarrow \infty} \left( \frac{1}{t} \right) \ln \left( \frac{\| y(t) \|}{\| y(0) \|} \right) . \quad (19)$$

where  $\| \cdot \|$  denotes a vector norm. The asymptotic quantity  $\lambda_i^t$  is called the Lyapunov Exponent.

If there exists a set of  $n$  Lyapunov Exponents associated with an  $n$ -dimensional autonomous system and they can be ordered by size that is

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots \geq \lambda_n . \quad (20)$$

The set of n-numbers  $\lambda_i$  is called the Lyapunov Spectrum.

Following Lyapunov[16], the fundamental matrix  $\Phi(t)$  is called regular if

$$\lim_{t \rightarrow \infty} \ln | \det \Phi(t) | \text{ .} \quad (21)$$

exist and is finite and if there exists a normal basis of the n-dimensional state space such that

$$\sum_{i=1}^n \lambda_i^t = \lim_{t \rightarrow \infty} \ln | \det \Phi(t) | \text{ .} \quad (22)$$

If  $\Phi(t)$  is regular, then according to a theorem Oseledec[2] the asymptotic quantity defined in (19) exists and is finite for any initial deviation  $y(0)$  belonging to the n-dimensional space.

The asymptotic quantity  $\lambda_i^t$ , given by (19) is also known as a one dimensional exponent. For  $p$ -dimensions, a  $p$ -dimensional Lyapunov exponent  $\lambda$  is defined as

$$\lambda^p = \lim_{t \rightarrow \infty} \left( \frac{1}{t} \right) \ln \left( \frac{\| y_1(t) \wedge y_2(t) \wedge \dots \wedge y_p(t) \|}{\| y_1(0) \wedge y_2(0) \wedge \dots \wedge y_p(0) \|} \right) \text{ .} \quad (23)$$

where  $\wedge$  is an exterior or vector cross product.

The above fact of Lyapunov Exponents is valid for any non-relativistic system. But in General Relativity, there is no concept of absolute time, therefore the time parameter forces us to consider equation (1) under spacetime diffeomorphism:  $u = \psi(x)$ ,  $d\tau = \eta(x)dt$ . As a result, chaos is a property of the physical system and does not depend on the coordinates used to describe the system. Therefore motivated by the work of Motter[1] chaos is characterized by positive Lyapunov Exponents. Similarly chaos is also characterized by positive Kolmogorov-Sinai Entropy ( $h_{ks}^\tau$ ). They are coordinate invariants and transform according to

$$\lambda_i^\tau = \frac{\lambda_i^t}{\langle \eta \rangle_t} \text{ .} \quad (24)$$

and

$$h_{ks}^\tau = \frac{h_{ks}^t}{\langle \eta \rangle_t} \text{ .} \quad (25)$$

where  $0 < \langle \eta \rangle_t < \infty$  is the time average of  $\eta = \frac{d\tau}{dt}$  over typical trajectory and  $i = 1, \dots, n$ ,  $n$  is the phase-space dimension. Transformation like  $u = \psi(x)$ ,  $d\tau = \eta(x)dt$  is composed of a time re-parametrization followed by a space diffeomorphism. It is well known that the Lyapunov Exponents and Kolmogorov-Sinai Entropy are invariant under space diffeomorphism[18].

### 3 Lyapunov Exponent and Effective Potential:

Now to compute the Lyapunov Exponent in terms of ISCO equation we shall first derive the radial potential in terms of Lyapunov Exponent. Therefore the Lagrangian for a test particle in the equatorial plane for any static spherically symmetric space-time can be written as

$$\mathcal{L} = \frac{1}{2} \left[ g_{tt} \dot{t}^2 + g_{rr} \dot{r}^2 + g_{\phi\phi} \dot{\phi}^2 \right] . \quad (26)$$

Now we defining the canonical momenta as

$$p_q = \frac{\delta \mathcal{L}}{\delta \dot{q}} . \quad (27)$$

Using it, the generalized momenta can be derived as

$$p_t = g_{tt} \dot{t} = -E = \text{Const} . \quad (28)$$

$$p_\phi = g_{\phi\phi} \dot{\phi} = L = \text{Const} . \quad (29)$$

$$p_r = g_{rr} \dot{r} . \quad (30)$$

Here  $(\dot{t}, \dot{r}, \dot{\phi})$  denotes differentiation with respect to proper time( $\tau$ ). Now from the Euler-Lagrange equations of motion

$$\frac{dp_q}{d\tau} = \frac{\delta \mathcal{L}}{\delta q} . \quad (31)$$

Using it we get the non-linear differential equation in 2-dimensional phase space with phase space variables  $X_i(t) = (p_r, r)$ .

$$\frac{dp_r}{d\tau} = \frac{\delta \mathcal{L}}{\delta r} . \quad (32)$$

and

$$\frac{dr}{d\tau} = \frac{p_r}{g_{rr}} . \quad (33)$$

Now linearizing the equations of motion about circular orbits of constant  $r$ , we get the infinitesimal evolution matrix as

$$M_{ij} = \left( \begin{array}{cc} 0 & \frac{d}{dr} \left( \frac{\delta \mathcal{L}}{\delta r} \right) \\ \frac{1}{g_{rr}} & 0 \end{array} \right) \Big|_{r=r_0} . \quad (34)$$

Now for circular orbits of constant  $r = r_0$  the characteristic values of the matrix gives the information about stability of the orbits. The eigen values of this matrix are principal

Lyapunov Exponent. Therefore the eigen values of the evolution matrix along circular orbits can be written as

$$\lambda^2 = \frac{1}{g_{rr}} \frac{d}{dr} \left( \frac{\delta \mathcal{L}}{\delta r} \right) \Big|_{r=r_0} . \quad (35)$$

Again from Lagrange's equation of motion

$$\frac{d}{d\lambda} \left( \frac{\delta \mathcal{L}}{\delta \dot{r}} \right) - \frac{\delta \mathcal{L}}{\delta r} = 0 . \quad (36)$$

Therefore the Lyapunov Exponent (which is the inverse of the instability time scale associated with the geodesic motions) in terms of the square of the radial velocity ( $\dot{r}^2$ ) can be written as

$$\frac{\delta \mathcal{L}}{\delta r} = \frac{1}{2g_{rr}} \frac{d}{dr} (\dot{r} g_{rr})^2 . \quad (37)$$

Finally the Principal Lyapunov Exponent can be rewritten as

$$\lambda^2 = \frac{1}{2} \frac{1}{g_{rr}} \frac{d}{dr} \left[ \frac{1}{g_{rr}} \frac{d}{dr} (\dot{r} g_{rr})^2 \right] . \quad (38)$$

Again for circular geodesics [17]

$$\dot{r}^2 = (\dot{r}^2)' = 0 . \quad (39)$$

where prime denotes for a derivative with respect to  $r$ . Therefore the equation (143) must be reduced to

$$\lambda^2 = \frac{(\dot{r}^2)''}{2} . \quad (40)$$

or

$$\lambda = \pm \sqrt{\frac{(\dot{r}^2)''}{2}} . \quad (41)$$

We shall show in the next section, in any spherically symmetric spacetime the radial equation can be written as in terms of the standard effective potential which is given by

$$\dot{r}^2 = E^2 - \mathcal{V}_{eff} . \quad (42)$$

Therefore the Principal Lyapunov Exponent can be expressed in terms of standard effective potential is

$$\lambda = \pm \sqrt{-\frac{\mathcal{V}_{eff}''}{2}} . \quad (43)$$



The Lyapunov Exponent must come in  $\pm$  pairs to conserve the volume of phase space. Here  $\mathcal{V}_{eff}'' > 0$  determines the stability and  $\mathcal{V}_{eff}'' < 0$  indicates instability[9]. Alternatively the circular orbit is unstable and chaotic when the  $\lambda$  is real, the circular orbit is stable when the  $\lambda$  is imaginary and the circular orbit is marginally stable when  $\lambda = 0$ .

The above expression for  $\lambda$  is valid for any stationary, spherically symmetric blackhole spacetimes i.e(Schwarzschild, Reissner Nordström, Schwarzschild-de Sitter, Schwarzschild-Anti-de Sitter, Reissner Nordström-de Sitter, Reissner Nordström-Anti de Sitter etc.).

## 4 Kolmogorov-Sinai Entropy and Lyapunov Exponent:

Another important quantity which is related to the Lyapunov Exponents is so called Kolmogorov-Sinai[3] Entropy ( $h_{ks}$ ), gives a measure of the amount of information lost or gained by a chaotic orbit as it evolves. Alternatively it determines how a system is chaotic or disorder when  $h_{ks} > 0$  and non-chaotic for  $h_{ks} = 0$  [18].

To compute it here, we use the formula which is given by Pesin[4] that it is equal to the sum of the positive Lyapunov Exponents i.e

$$h_{ks} = \sum_{\lambda_i > 0} \lambda_i . \quad (44)$$

In 2-dimensional phase-space, there are two Lyapunov Exponent, since  $h_{ks}$  is equal to the sum of positive Lyapunov Exponent, therefore here the Kolmogorov-Sinai entropy in terms of effective potential is given by

$$h_{ks} = \sqrt{-\frac{\mathcal{V}_{eff}''}{2}} . \quad (45)$$

This entropy have played a crucial role in mathematical theory of chaos to check whether a trajectory in dynamical system is in disorder or not when it evolves with time. It is some sense different from the physical entropy, for example the entropy of the 2nd law of thermodynamics or blackhole entropy. Formally it is defined somewhat like entropy in statistical mechanics i.e it involves a partition of phase space.

### 4.1 Critical Exponent and Effective Potential:

Following Pretorius and Khurana[12], we can define a Critical exponent which is the ratio of Lyapunov time scale  $T_\lambda$  and Orbital time scale  $T_\Omega$  can be written as

$$\gamma = \frac{\Omega}{2\pi\lambda} = \frac{T_\lambda}{T_\Omega} = \frac{Lyapunov\ TimeScale}{Orbital\ TimeScale} . \quad (46)$$

where we have introduced  $T_\lambda = \frac{1}{\lambda}$  and  $T_\omega = \frac{2\pi}{\Omega}$ , which is important for black-hole merger in the ring down radiation. In terms of the square of the radial velocity ( $\dot{r}^2$ ), Critical Exponent can be written as

$$\gamma = \frac{T_\lambda}{T_\Omega} = \frac{1}{2\pi} \sqrt{\frac{2\Omega^2}{(\dot{r}^2)''}} . \quad (47)$$

Now this exponent can be written as in terms of the inverse of the effective potential is given by

$$\gamma = \frac{T_\lambda}{T_\Omega} = \frac{1}{2\pi} \sqrt{-\frac{2\Omega^2}{\mathcal{V}_{eff}''}} . \quad (48)$$

Alternatively the reciprocal of critical exponent is proportional to the effective potential which is given by

$$\frac{1}{\gamma} = \frac{T_\Omega}{T_\lambda} = 2\pi \sqrt{-\frac{\mathcal{V}_{eff}''}{2\Omega^2}} . \quad (49)$$

## 5 Equatorial Circular Geodesics in Reissner Nordström Spacetime:

First, we shall consider a static, spherically symmetric, asymptotically flat solution of the coupled Einstein-Maxwell equations which represented by the Reissner Nordström spacetime, the metric for such space-time becomes

$$ds^2 = -\frac{\Delta}{r^2} dt^2 + \frac{r^2}{\Delta} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) . \quad (50)$$

where  $\Delta = r^2 - 2Mr + Q^2$ . For  $M > Q$ , the horizons are situated at  $r_\pm = M \pm \sqrt{M^2 - Q^2}$ , for  $M = Q$ , the horizon is at  $r = M$  and for  $M < Q$ , there is no horizon, the singularity at  $r = 0$  is naked singularity.

To compute the geodesics in the equatorial plane for the Reissner Nordström spacetime, we follow ([17], [7]). To determine the geodesic motions of a test particle in this plane we set  $\dot{\theta} = 0$  and  $\theta = \text{constant} = \frac{\pi}{2}$ .

Therefore the necessary Lagrangian for this motion is given by

$$\mathcal{L} = \frac{1}{2} \left[ -\frac{\Delta}{r^2} \dot{t}^2 + \frac{r^2}{\Delta} \dot{r}^2 + r^2 \dot{\phi}^2 \right] . \quad (51)$$

Now we defining the canonical momenta as

$$p_q = \frac{\delta \mathcal{L}}{\delta \dot{q}} . \quad (52)$$

Using it, the generalized momenta can be derived as

$$p_t = -\frac{\Delta}{r^2} \dot{t} = -E = \text{Const} . \quad (53)$$

$$p_\phi = r^2 \dot{\phi} = L = \text{Const} . \quad (54)$$

$$p_r = \frac{r^2}{\Delta} \dot{r} . \quad (55)$$

Here over dot denotes differentiation with respect to proper time( $\tau$ ). Since the Lagrangian does not depends on 't' and ' $\phi$ ', so  $p_t$  and  $p_\phi$  are conserved quantities. Solving (53) and (54) for  $\dot{t}$  and  $\dot{\phi}$  we find

$$\dot{t} = E \frac{r^2}{\Delta} . \quad (56)$$

$$\dot{\phi} = L/r^2 . \quad (57)$$

where  $E$  and  $L$  are the energy and angular momentum per unit mass of the test particle. Therefore the necessary Hamiltonian is given by

$$\mathcal{H} = p_t \dot{t} + p_\phi \dot{\phi} + p_r \dot{r} - \mathcal{L} . \quad (58)$$

In terms of the metric the Hamiltonian is

$$\mathcal{H} = -\frac{\Delta}{r^2} \dot{t}^2 + \frac{r^2}{\Delta} \dot{r}^2 + r^2 \dot{\phi}^2 - \mathcal{L} . \quad (59)$$

Since the Hamiltonian is independent of 't', therefore we can write it as

$$2\mathcal{H} = -\frac{\Delta}{r^2} \dot{t}^2 + \frac{r^2}{\Delta} \dot{r}^2 + r^2 \dot{\phi}^2 . \quad (60)$$

$$= -E \dot{t} + L \dot{\phi} + \frac{r^2}{\Delta} \dot{r}^2 = \epsilon = \text{const} . \quad (61)$$

Here  $\epsilon = -1$  for time-like geodesics,  $\epsilon = 0$  for light-like geodesics and  $\epsilon = +1$  for space-like geodesics. Substituting the equations (56) and (57) in (61), we obtain the radial equation for any spherically space-time is given by

$$\dot{r}^2 = E^2 - \mathcal{V}_{eff} = E^2 - \left( \frac{L^2}{r^2} - \epsilon \right) \frac{\Delta}{r^2} . \quad (62)$$

where the standard effective potential for Reissner Nordström spacetime is

$$\mathcal{V}_{eff} = \left( \frac{L^2}{r^2} - \epsilon \right) \frac{\Delta}{r^2} . \quad (63)$$

To investigate the circular geodesic motions of the test particle in the Einstein -Maxwell gravitational field , we must have for circular geodesics of constant  $r = r_0$  and from the equation (62) we get

$$\mathcal{V}_{eff} = E^2 . \quad (64)$$

and

$$\frac{d\mathcal{V}_{eff}}{dr} = 0 . \quad (65)$$

Therefore we obtain the energy and angular momentum per unit mass of the test particle are given by

$$E^2 = -\frac{(r_0^2 - 2Mr_0 + Q^2)^2}{r_0^2(r_0^2 - 3Mr_0 + 2Q^2)}\epsilon . \quad (66)$$

and

$$L^2 = -\frac{r_0^2(Mr_0 - Q^2)}{r_0^2 - 3Mr_0 + 2Q^2}\epsilon . \quad (67)$$

## 6 Non-Extremal Reissner Nordström Spacetime

### 6.1 Lyapunov Exponent and Equation of ISCO:

For non-extremal case the radial potential of the test particle for time-like circular geodesics by substituting  $\epsilon = -1$  is given by

$$\mathcal{V}_{eff} = \left( 1 + \frac{L^2}{r^2} \right) \left( 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \right) . \quad (68)$$

and correspondingly the value of energy  $E$  and angular momentum associated with the circular orbit at  $r = r_0$  are given by

$$E^2 = \frac{(r_0^2 - 2Mr_0 + Q^2)^2}{r_0^2 - 3Mr_0 + 2Q^2} . \quad (69)$$

$$L^2 = \frac{r_0^2(Mr_0 - Q^2)}{r_0^2 - 3Mr_0 + 2Q^2} . \quad (70)$$

Circular motion of the test particle to be exists when both energy and angular momentum are real and finite, therefore we must have  $r_0^2 - 3Mr_0 + 2Q^2 > 0$  and  $r_0 > \frac{Q^2}{M}$ .

Also the angular frequency measured by an asymptotic observers for time-like circular geodesics at  $r = r_0$  is given by  $\Omega_0$

$$\Omega_0 = \frac{\dot{\phi}}{\dot{t}} = \frac{\sqrt{(Mr_0 - Q^2)}}{r_0^2} . \quad (71)$$

Therefore for non-extremal Reissner Nordström black-hole, the Lyapunov Exponents in terms of the ISCO equation becomes

$$\lambda_{RN} = \sqrt{\frac{-(Mr_0^3 - 6M^2r_0^2 + 9MQ^2r_0 - 4Q^4)}{r_0^4(r_0^2 - 3Mr_0 + 2Q^2)}} . \quad (72)$$

and the Kolmogorov-Sinai Entropy for non-extremal Reissner Nordström black-hole becomes

$$h_{ks} = \sqrt{\frac{-(Mr_0^3 - 6M^2r_0^2 + 9MQ^2r_0 - 4Q^4)}{r_0^4(r_0^2 - 3Mr_0 + 2Q^2)}} . \quad (73)$$

So the time like circular geodesics of non-extremal Reissner Nordström black-hole are stable when

$$Mr_0^3 - 6M^2r_0^2 + 9MQ^2r_0 - 4Q^4 > 0 . \quad (74)$$

such that  $\lambda_{RN}$  is imaginary, the circular geodesics are unstable when

$$Mr_0^3 - 6M^2r_0^2 + 9MQ^2r_0 - 4Q^4 < 0 . \quad (75)$$

i.e  $\lambda_{RN}$  is real and the time-like circular geodesics is marginally stable when

$$Mr_0^3 - 6M^2r_0^2 + 9MQ^2r_0 - 4Q^4 = 0 . \quad (76)$$

such that  $\lambda_{RN}$  is zero.

The circular orbits are chaotic around the non-extremal Reissner Nordström black-hole for  $Mr_0^3 - 6M^2r_0^2 + 9MQ^2r_0 - 4Q^4 < 0$  i.e.  $h_{ks}$  is real and for ISCO,  $h_{ks}$  is zero.

The solution of the preceding equation gives the radius of ISCO at  $r_0 = r_{ISCO}$  for non-extremal Reissner-Nordström black-hole[11] which is given by

$$\frac{r_{ISCO}}{M} = 2 + Z_2 + \frac{Z_3}{Z_2} . \quad (77)$$

where

$$\begin{aligned} Z_3 &= 4 - 3\left(\frac{Q^2}{M^2}\right) \\ Z_2 &= \left(8 + 2\frac{Q^4}{M^4} + \frac{Q^2}{M^2}Z_1\right)^{1/3} \\ Z_1 &= \left[-9 + \sqrt{5 - 9\frac{Q^2}{M^2} + 4\frac{Q^4}{M^4}}\right] \end{aligned}$$

In the limit  $Q = 0$ ,  $Z_3 = 4$  and  $Z_2 = 2$ , we get the radius of ISCO,  $r_{ISCO} = 6M$  for Schwarzschild black-hole.

Now the reciprocal of critical exponent in terms of ISCO equation for Reissner-Nordström blackhole is given by

$$\frac{1}{\gamma} = \frac{T_\Omega}{T_\lambda} = 2\pi \sqrt{\frac{-(Mr_0^3 - 6M^2r_0^2 + 9MQ^2r_0 - 4Q^4)}{(r_0^2 - 3Mr_0 + 2Q^2)}}. \quad (78)$$

For any unstable circular orbit,  $T_\Omega > T_\lambda$  i.e. Lyapunov time scale is shorter than the gravitational time scale making the instability observationally relevant[6].

### 6.1.1 Schwarzschild Black-hole:

For Schwarzschild black hole  $a = Q = 0$ , the Lyapunov Exponents in terms of ISCO equation are

$$\lambda_{Sch} = \sqrt{-\frac{M(r_0 - 6M)}{r_0^3(r_0 - 3M)}}. \quad (79)$$

and the Kolmogorov-Sinai Entropy for Schwarzschild blackhole becomes

$$h_{ks} = \sqrt{-\frac{M(r_0 - 6M)}{r_0^3(r_0 - 3M)}}. \quad (80)$$

So the time like circular geodesics of Schwarzschild black-hole are stable when

$$r_0 - 6M > 0. \quad (81)$$

such that  $\lambda_{Sch}$  or  $h_{ks}$  is imaginary, the circular geodesics are unstable when

$$3M < r_0 < 6M. \quad (82)$$

i.e  $\lambda_{Sch}$  or  $h_{ks}$  is real and the time like circular geodesics is marginally stable when

$$r_0 - 6M = 0 . \quad (83)$$

such that  $\lambda_{Sch}$  or  $h_{ks}$  is zero. This radius at  $r_0 = r_{ISCO} = 6M$  gives the ISCO for Schwarzschild blackhole. The critical exponent for Schwarzschild blackhole is given by

$$\frac{1}{\gamma} = \frac{T_\Omega}{T_\lambda} = 2\pi \sqrt{\frac{-(r_0 - 6M)}{(r_0 - 3M)}} . \quad (84)$$

For any unstable circular orbit say for  $r_0 = 4M$ ,  $\gamma$  becomes  $\frac{1}{2\sqrt{2}\pi}$ , therefore  $T_\lambda < T_\Omega$ , i.e. Lyapunov time scale is shorter than the gravitational time scale, for this case the chaos will be damped and it is unobservable[6].

## 6.2 Lyapunov Exponent and Null Circular Geodesics:

The radial potential that governs the null geodesics, can be expressed as by substituting  $\epsilon = 0$  in (62)

$$\dot{r}^2 = E^2 - \mathcal{U}_{eff} = E^2 - \frac{L^2}{r^2} \left( 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \right) . \quad (85)$$

where the effective potential for null geodesics is given by

$$\mathcal{U}_{eff} = \frac{L^2}{r^2} \frac{\Delta}{r^2} . \quad (86)$$

For circular null geodesics at  $r = r_c$ , we know the condition from (39)

$$\mathcal{U}_{eff} = E^2 . \quad (87)$$

and

$$\frac{d\mathcal{U}_{eff}}{dr} = 0 . \quad (88)$$

Therefore we obtain the ratio of energy and angular momentum of the test particle evaluated at  $r = r_c$  for circular null geodesics are given by

$$\frac{E}{L} = \pm \sqrt{\frac{r_c^2 - 2Mr_c + Q^2}{r_c^4}} . \quad (89)$$

$$r_c^2 - 3Mr_c + 2Q^2 = 0 . \quad (90)$$

After introducing the impact parameter  $D_c = \frac{L}{E}$ , the above equations reduced to

$$\frac{1}{D_c} = \frac{E}{L} = \sqrt{\frac{Mr_c - Q^2}{r_c^4}}. \quad (91)$$

Solving equation (90) we obtain the radius of null circular orbits are

$$r_{c\pm} = \frac{3M}{2} \left[ 1 \pm \sqrt{1 - \frac{8}{9} \left( \frac{Q^2}{M^2} \right)} \right]. \quad (92)$$

It can be easily seen that from equation (90) circular null geodesics is the limiting case of the timelike circular geodesics. Therefore for this case the 2nd derivative of the effective potential is

$$\mathcal{U}_{eff}'' = \frac{L^2}{r_c^4} \left( 1 - 2 \frac{Q^2}{r_c^2} \right). \quad (93)$$

Again the angular frequency measured by an asymptotic observers which is given by

$$\Omega_c = \frac{\dot{\phi}}{\dot{t}} = \frac{\Delta}{r_c^4} D_c = \frac{1}{D_c} = \sqrt{\frac{Mr_c - Q^2}{r_c^4}}. \quad (94)$$

Using equation (91) we show that the angular frequency  $\Omega_c$  of the circular null geodesics is inverse of the impact parameter  $D_c$ .

Using (41) the Lyapunov Exponent for null circular geodesics is given by

$$\lambda_{RNN} = \sqrt{\frac{L^2}{r_c^4}} \sqrt{1 - \frac{2Q^2}{r_c^2}}. \quad (95)$$

and the Kolmogorov-Sinai Entropy for Reissner Nordström blackhole becomes

$$h_{ks} = \sqrt{\frac{L^2}{r_c^4}} \sqrt{1 - \frac{2Q^2}{r_c^2}}. \quad (96)$$

So the circular geodesics  $r_c = r_{c+}$  and  $r_c = r_{c-}$  are unstable since  $\lambda_{RNN}$  is real.

### 6.2.1 Schwarzschild Black-hole:

For Schwarzschild black hole  $a = Q = 0$ , and the Lyapunov Exponents are

$$\lambda_{SchN} = \sqrt{\frac{L^2}{r_c^4}}. \quad (97)$$

It can be easily check that for  $r_c = 3M$ ,  $\lambda_{SchN}$  is real, so the Schwarzschild photon sphere is unstable.



## 7 Extremal Reissner Nordström Spacetime

### 7.1 Lyapunov Exponent and Equation of ISCO:

To derive the Lyapunov Exponent in terms of the equation of ISCO, we shall first write the effective potential for timelike circular geodesics which is given by

$$\mathcal{V}_{eff} = \left(1 + \frac{L^2}{r^2}\right) \left(1 - \frac{M}{r}\right)^2. \quad (98)$$

Therefore we obtain readily the value of energy  $E$  associated with the circular orbit is given by

$$E^2 = \frac{(r_0 - M)^3}{r_0^2(r_0 - 2M)}. \quad (99)$$

and the value of angular momentum associated with the circular orbit is given by

$$L^2 = \frac{Mr_0^2}{(r_0 - 2M)}. \quad (100)$$

For circular motions, it must be required that energy  $E$  and angular momentum  $L$ , which are both real and finite, so we must have  $r_0 - 2M > 0$ . Again the angular frequency measured by an asymptotic observers for time-like circular geodesics is given by  $\Omega$

$$\Omega_0 = \frac{\dot{\phi}}{\dot{t}} = \frac{\sqrt{M(r_0 - M)}}{r_0^2}. \quad (101)$$

Therefore for extremal Reissner Nordström black-hole, the Lyapunov Exponents in terms of the ISCO equation becomes

$$\lambda_{exRN} = \frac{\sqrt{M(r_0 - M)}}{r_0^2 \sqrt{(r_0 - 2M)}} \sqrt{-(r_0 - 4M)}. \quad (102)$$

and the Kolmogorov-Sinai Entropy for extremal Reissner Nordström black-hole becomes

$$h_{ks} = \frac{\sqrt{M(r_0 - M)}}{r_0^2 \sqrt{(r_0 - 2M)}} \sqrt{-(r_0 - 4M)}. \quad (103)$$

So the time like circular geodesics of extremal Reissner Nordström black-hole are stable when

$$r_0 - 4M > 0. \quad (104)$$

such that  $\lambda_{exRN}$  is imaginary, the circular geodesics are unstable when

$$2M < r_0 < 4M . \quad (105)$$

i.e  $\lambda_{exRN}$  is real and the time-like circular geodesics is marginally stable when

$$r_0 - 4M = 0 . \quad (106)$$

such that  $\lambda_{exRN}$  is zero.

Again the calculation for Kolmogorov-Sinai Entropy  $h_{ks}$  for extremal Reissner Nordström black-hole is valid for  $r_0 \neq r_c$ . For  $r_0 = r_c = M$ , the  $h_{ks}$  becomes zero.

## 7.2 Lyapunov Exponent and Null Circular Geodesics:

The effective potential that governs the null circular geodesics, can be expressed as by substituting  $\epsilon = 0$  in (62)

$$\mathcal{U}_{eff} = \frac{L^2}{r^2} \left(1 - \frac{M}{r}\right)^2 . \quad (107)$$

Therefore the ratio of energy and angular momentum of the test particle evaluated at  $r = r_c$  for circular null geodesics are given by

$$\frac{E}{L} = \pm \frac{(r_c - M)}{r_c^2} . \quad (108)$$

$$r_c^2 - 3Mr_c + 2M^2 = 0 . \quad (109)$$

After introducing the impact parameter  $D_c = \frac{L}{E}$ , the above equations reduced to

$$\frac{1}{D_c} = \frac{E}{L} = \sqrt{\frac{M(r_c - M)}{r_c^4}} . \quad (110)$$

Solving equation (109) we obtain the radius of null circular orbits are

$$r_c = M, 2M . \quad (111)$$

It can be easily seen that from equation (111) circular null geodesics is the limiting case of the timelike circular geodesics. Therefore for this case the 2nd derivative of the effective potential is

$$\mathcal{U}_{eff}'' = \frac{L^2}{r_c^4} \left(1 - 2\frac{M^2}{r_c^2}\right) . \quad (112)$$

Again the angular frequency measured by an asymptotic observers which is given by

$$\Omega_c = \frac{\dot{\phi}}{\dot{t}} = \frac{(r_c - M)^4}{r_c^4} D_c = \frac{1}{D_c} = \sqrt{\frac{M(r_c - M)}{r_c^4}}. \quad (113)$$

Using equation (110) we show that the angular frequency  $\Omega_c$  of the circular null geodesics is inverse of the impact parameter  $D_c$ .

Using (41) the Lyapunov Exponent for null circular geodesics is given by

$$\lambda_{exRNN} = \sqrt{\frac{L^2}{r_c^4} \left(1 - \frac{2M^2}{r_c^2}\right)}. \quad (114)$$

So the circular geodesics  $r_c = 2M$  are unstable since  $\lambda_{exRNN}$  is real.

*Note that* for extremal blackhole, this result is valid for only single null geodesics i.e.  $r_0 \neq r_c$ . For  $r_0 = r_c = M$ , the Lyapunov Exponent becomes zero i.e.  $\lambda_{exRN} = \lambda_{exRNN} = 0$ .

Now we shall make a link between geodesic deviation equation and Lyapunov Exponent.

## 8 Geodesic Deviation Equation and Lyapunov Exponent

Geodesic deviation equation gives a proper measure of spacetime curvature. It connects the acceleration of the separation vector  $\chi^\mu$  between two nearly geodesics. The separation four vector  $\chi(\tau)$  connects a point  $x^\alpha(\tau)$  on one geodesic(the fiducial geodesic) to a point  $x^\alpha(\tau) + \chi^\alpha(\tau)$  on a nearby geodesic at the same proper time. In this section we shall find the link between Geodesic Deviation Equation and Lyapunov Exponent.

### 8.1 Timelike Case:

We know the timelike geodesic equation is given by

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0 \quad (115)$$

where  $\tau$  is proper time along the geodesics and  $\frac{dx^\alpha}{d\tau}$  is the velocity four vector. Now to evaluate the geodesic deviation equation,  $x^\alpha(\tau) + \chi^\alpha(\tau)$  must obey the geodesic equation (115). Therefore we obtain the geodesic deviation equation which is given by

$$\frac{d^2 \chi^\mu}{d\tau^2} + 2\Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{d\chi^\beta}{d\tau} + \Gamma^\mu_{\alpha\beta,\delta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} \chi^\delta = 0 \quad (116)$$

where  $\chi^\delta$  is the deviation four vector. We know the spherically symmetric RN spacetime can be written as

$$ds^2 = -g(r)dt^2 + \frac{1}{g(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (117)$$

where  $g(r) = 1 - 2M/r + Q^2/r^2$ .

Using (115) and (117) we get the following equations for timelike geodesics:

$$\frac{d^2t}{d\tau^2} = -\frac{g'(r)}{g(r)} \frac{dt}{d\tau} \frac{dr}{d\tau} \quad (118)$$

$$\frac{d^2r}{d\tau^2} = -\frac{1}{2}g(r)g'(r)\left(\frac{dt}{d\tau}\right)^2 + \frac{g'(r)}{2g(r)}\left(\frac{dr}{d\tau}\right)^2 + rg(r)\left(\frac{d\theta}{d\tau}\right)^2 + rg(r)\sin^2\theta\left(\frac{d\phi}{d\tau}\right)^2 \quad (119)$$

$$\frac{d^2\theta}{d\tau^2} = -\frac{2}{r}\frac{dr}{d\tau}\frac{d\theta}{d\tau} + \sin\theta\cos\theta\left(\frac{d\phi}{d\tau}\right)^2 \quad (120)$$

$$\frac{d^2\phi}{d\tau^2} = -\frac{2}{r}\frac{dr}{d\tau}\frac{d\phi}{d\tau} - 2\cot\theta\frac{d\theta}{d\tau}\frac{d\phi}{d\tau} \quad (121)$$

Hence for circular geodesics of constant  $r = r_0$  on the equatorial plane  $\theta = \pi/2$ , these equations reduces to

$$\frac{d^2t}{d\tau^2} = 0 \quad (122)$$

$$\frac{1}{2}g'(r_0)\left(\frac{dt}{d\tau}\right)^2 - r_0\left(\frac{d\phi}{d\tau}\right)^2 = 0 \quad (123)$$

$$\frac{d^2\theta}{d\tau^2} = 0 \quad (124)$$

$$\frac{d^2\phi}{d\tau^2} = 0 \quad (125)$$

and, using (116) and (117) we obtain following equations for geodesic deviation equation

$$\frac{d^2\chi^t}{d\tau^2} + \frac{g'(r_0)}{g(r_0)}\left(\frac{dt}{d\tau}\right)\left(\frac{d\chi^r}{d\tau}\right) = 0 \quad (126)$$

$$\begin{aligned} & \frac{d^2\chi^r}{d\tau^2} + [g(r_0)g'(r_0)\left(\frac{dt}{d\tau}\right)\left(\frac{d\chi^t}{d\tau}\right) - 2r_0g(r_0)\left(\frac{d\phi}{d\tau}\right)\left(\frac{d\chi^\phi}{d\tau}\right)] + \\ & \left[\frac{1}{2}\{(g'(r_0))^2 + g(r_0)g''(r_0)\}\left(\frac{dt}{d\tau}\right)^2 - \{g(r_0) + rg'(r_0)\}\left(\frac{d\phi}{d\tau}\right)^2\right]\chi^r = 0 \end{aligned} \quad (127)$$

$$\frac{d^2\chi^\theta}{d\tau^2} + \left(\frac{d\phi}{d\tau}\right)^2\chi^\theta = 0 \quad (128)$$

$$\frac{d^2\chi^\phi}{d\tau^2} + \frac{2}{r_0}\left(\frac{d\phi}{d\tau}\right)\left(\frac{d\chi^r}{d\tau}\right) = 0 \quad (129)$$

Again for timelike circular geodesics  $u.u = -1$  Therefore using (117) we get

$$g(r_0)\left(\frac{dt}{d\tau}\right)^2 - r_0^2\left(\frac{d\phi}{d\tau}\right)^2 = 0 \quad (130)$$

Solving equations (123) and (130) we have

$$\left(\frac{d\phi}{d\tau}\right)^2 = \frac{g'(r_0)}{r[2g(r_0) - r_0g'(r_0)]} \quad (131)$$

$$\left(\frac{dt}{d\tau}\right)^2 = \frac{2}{2g(r_0) - r_0g'(r_0)} \quad (132)$$

Using (131) and (126-(129)) we transform the equation into  $(r - \phi)$  plane we get the following equations

$$\frac{d^2\chi^t}{d\phi^2} + \frac{g'(r_0)}{g(r_0)}\left(\frac{dt}{d\phi}\right)\left(\frac{d\chi^r}{d\phi}\right) = 0 \quad (133)$$

$$\begin{aligned} & \frac{d^2\chi^r}{d\phi^2} + [g(r_0)g'(r_0)\left(\frac{dt}{d\phi}\right)\left(\frac{d\chi^t}{d\phi}\right) - 2r_0g(r_0)\left(\frac{d\chi^\phi}{d\phi}\right)] + \\ & \left[\frac{1}{2}\{(g'(r_0))^2 + g(r_0)g''(r_0)\}\left(\frac{dt}{d\phi}\right)^2 - \{g(r_0) + r_0g'(r_0)\}\right]\chi^r = 0 \end{aligned} \quad (134)$$

$$\frac{d^2\chi^\theta}{d\phi^2} + \chi^\theta = 0 \quad (135)$$

$$\frac{d^2\chi^\phi}{d\phi^2} + \frac{2}{r_0}\frac{d\chi^r}{d\phi} = 0 \quad (136)$$

From the equation (135), we see that it represents a S.H.M, this implies that the motion in the equatorial plane  $\theta = \pi/2$  is stable. Since the motion is harmonic, so we assume that the trial solutions of the remaining equations are

$$\chi^t = c_1 e^{i\omega\phi} \quad (137)$$

$$\chi^r = c_2 e^{i\omega\phi} \quad (138)$$

$$\chi^\phi = c_3 e^{i\omega\phi} \quad (139)$$

we get the reduced radial deviation equation is given by

$$\frac{d^2\chi^r}{d\phi^2} + \left( \frac{g(r_0)g'(r_0) + r_0 \frac{3g(r_0)g'(r_0)}{r_0} - 2(g'(r_0))^2}{g'(r_0)} r_0 \right) \chi^r = 0 \quad (140)$$

Again for any static, spherically symmetric metric like (117) the radial equation that governs the radial motions of the test particle is given by

$$\dot{r}^2 = E^2 - \mathcal{V}_{eff} = E^2 - \left( \frac{L^2}{r^2} - \epsilon \right) g(r) . \quad (141)$$

where the standard effective potential for any spherically symmetric spacetime (Schwarzschild, Reissner Nordström) is

$$\mathcal{V}_{eff} = \left( \frac{L^2}{r^2} - \epsilon \right) g(r) . \quad (142)$$

and  $\epsilon$  has been previously defined. Once again we know for circular geodesics say  $r = r_0$

$$\mathcal{V}_{eff} = E^2 . \quad (143)$$

and

$$\frac{d\mathcal{V}_{eff}}{dr} = 0 . \quad (144)$$

Therefore we obtain the energy and angular momentum per unit mass of the test particle for timelike orbit are given by

$$E^2 = \frac{2g(r_0)^2}{2g(r_0) - rg'(r_0)} . \quad (145)$$

and

$$L^2 = \frac{r_0^3 g'(r_0)}{2g(r_0) - r_0 g'(r_0)} . \quad (146)$$

Therefore the 2nd derivative of effective potential (142) for timelike circular geodesics ( $\epsilon = -1$ ) of constant  $r = r_0$  is given by

$$\mathcal{V}_{eff}'' = 2 \frac{\left[ g(r_0)g'(r_0) + \frac{3g(r_0)g'(r_0)}{r_0} - 2(g'(r_0))^2 \right]}{2g(r_0) - r_0 g'(r_0)} \quad (147)$$

and the Lyapunov Exponent in terms of effective potential for any spherically symmetry spacetime is given by

$$\lambda_{sph} = \sqrt{-\frac{\mathcal{V}_{eff}''}{2}}. \quad (148)$$

Using (140) and (147) we get the following equation in terms of the effective potential

$$\frac{d^2\chi^r}{d\phi^2} + \left[ \frac{r_0(2g(r_0) - r_0g'(r_0))}{2g'(r_0)} \right] \mathcal{V}_{eff}'' \chi^r = 0 \quad (149)$$

Using (140), (147) and (148) we get the radial geodesic deviation equation *in terms of* the Lyapunov Exponent is given by

$$\frac{d^2\chi^r}{d\phi^2} - \frac{r_0(2g(r_0) - r_0g'(r_0))}{2g'(r_0)} \lambda_{sph}^2 \chi^r = 0 \quad (150)$$

where the Lyapunov Exponent for any spherically symmetry blackhole is given by

$$\lambda_{sph} = \sqrt{-\frac{[g(r_0)g'(r_0) + \frac{3g(r_0)g'(r_0)}{r_0} - 2(g'(r_0))^2]}{2g(r_0) - r_0g'(r_0)}}. \quad (151)$$

Since  $2g(r_0) - r_0g'(r_0) > 0$  and  $g'(r_0) > 0$  for the allowed geodesic motion in the equatorial plane, so the solution of the equation is unstable for  $\lambda_{sph}^2 > 0$ , stable  $\lambda_{sph}^2 < 0$  and marginal stable for  $\lambda_{sph}^2 = 0$ .

For Reissner Nordström blackhole  $g(r_0) = 1 - 2M/r_0 + Q^2/r_0^2$ ,  $2g(r_0) - r_0g'(r_0) = r_0^2 - 3Mr_0 + 2Q^2$  and  $g'(r_0) = 2(Mr_0 - Q^2)/r_0^3$ , therefore the Lyapunov Exponent becomes

$$\lambda_{RN} = \sqrt{\frac{-(Mr_0^3 - 6M^2r_0^2 + 9MQ^2r_0 - 4Q^4)}{r_0^4(r_0^2 - 3Mr_0 + 2Q^2)}}. \quad (152)$$

which is similar to the expression (72) and correspondingly the geodesic deviation equation is

$$\frac{d^2\chi^r}{d\phi^2} + \frac{r_0(Mr_0^3 - 6M^2r_0^2 + 9MQ^2r_0 - 4Q^4)}{2(Mr_0 - Q^2)} \chi^r = 0 \quad (153)$$

For Schwarzschild blackhole  $g(r_0) = 1 - 2M/r_0$ ,  $2g(r_0) - r_0g'(r_0) = r_0^2 - 3Mr_0$  and  $g'(r_0) = 2M/r_0^2$ , therefore the Lyapunov Exponent becomes

$$\lambda_{Sch} = \sqrt{\frac{-M(r_0 - 6M)}{r_0^3(r_0 - 3M)}}. \quad (154)$$

which is similar to the expression (83) and correspondingly the geodesic deviation equation is

$$\frac{d^2\chi^r}{d\phi^2} + \frac{r_0(r_0 - 6M)}{2} \chi^r = 0 \quad (155)$$

## 8.2 Null Case:

For null circular orbit in the equatorial plane, the geodesic deviation is given by

$$\frac{d^2\chi^\mu}{d\lambda^2} + 2\Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{d\chi^\beta}{d\lambda} + \Gamma^\mu_{\alpha\beta,\delta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} \chi^\delta = 0 \quad (156)$$

Analogously for null circular orbit in the equatorial plane, the geodesic deviation equation is given by

$$\frac{d^2\chi^r}{d\phi^2} + \frac{1}{2}[r_c^2 g''(r_c) - 2g(r_c)] \chi^r = 0 \quad (157)$$

Again from (142) the effective potential for null geodesics is given by

$$\mathcal{U}_{eff} = \frac{L^2}{r^2} g(r) . \quad (158)$$

For circular null geodesics at  $r = r_c$

$$\mathcal{U}_{eff} = E^2 . \quad (159)$$

and

$$\frac{d\mathcal{U}_{eff}}{dr} = 0 . \quad (160)$$

Therefore we obtain the ratio of energy and angular momentum of the test particle evaluated at  $r = r_c$  for circular null geodesics are given by

$$\frac{E}{L} = \pm \sqrt{\frac{g(r_c)}{r_c^2}} . \quad (161)$$

$$2g(r_c) - r_c g'(r_c) = 0 . \quad (162)$$

It can be easily seen that from equation (162) circular null geodesics is the approximation of the timelike circular geodesics.

Therefore for this case the 2nd derivative of the effective potential is

$$\mathcal{U}_{eff}'' = \frac{L^2}{r_c^4} (r_c^2 g''(r_c) - 2g(r_c)) . \quad (163)$$

Thus the Lyapunov Exponent in terms of effective potential for null circular geodesics can be written as

$$\lambda_{Null} = \sqrt{-\frac{\mathcal{U}_{eff}''}{2}} . \quad (164)$$



Using (157) and (147) we get the following equation in terms of the radial potential becomes

$$\frac{d^2\chi^r}{d\phi^2} + \left( \frac{r^4}{2L^2} \mathcal{U}_{eff}'' \right) |_{r=r_c} \chi^r = 0 \quad (165)$$

Using (157), (147) and (164) we get the radial geodesic deviation equation *in terms of* the Lyapunov Exponent for null case is given by

$$\frac{d^2\chi^r}{d\phi^2} - \left( \frac{r_c^4}{L^2} \lambda_{Null}^2 \right) \chi^r = 0 \quad (166)$$

where the Lyapunov Exponent for null circular geodesics is given by

$$\lambda_{Null} = \sqrt{\frac{L^2}{2r_c^4} [2g(r_c) - r_c^2 g''(r_c)]} . \quad (167)$$

The solution of the equation is unstable for  $\lambda_{Null}^2 > 0$ , stable  $\lambda_{Null}^2 < 0$  and marginal stable for  $\lambda_{Null}^2 = 0$  for the allowed geodesic motions in the equatorial plane.

For Reissner Nordström blackhole  $g(r_c) = 1 - 2M/r_c + Q^2/r_c^2$  and  $2g(r_c) - r_c^2 g''(r_c) = 2(1 - 2\frac{Q^2}{r_c^2})$  therefore the Lyapunov Exponent becomes

$$\lambda_{Null} = \sqrt{\frac{L^2}{r_c^4} \left( 1 - 2\frac{Q^2}{r_c^2} \right)} . \quad (168)$$

which is similar to the expression (95) and correspondingly the geodesic deviation equation is

$$\frac{d^2\chi^r}{d\phi^2} - \left( 1 - 2\frac{Q^2}{r_c^2} \right) \chi^r = 0 \quad (169)$$

For Schwarzschild blackhole  $g(r_c) = 1 - 2M/r_0$ ,  $2g(r_c) - r_c^2 g''(r_c) = 2$  therefore the Lyapunov Exponent becomes

$$\lambda_{Null} = \sqrt{\frac{L^2}{r_c^4}} . \quad (170)$$

which is similar to the expression (97) and correspondingly the geodesic deviation equation is

$$\frac{d^2\chi^r}{d\phi^2} - \chi^r = 0 \quad (171)$$

## 9 Ratio of Angular velocity of time like circular orbit to Null Circular Orbit

Since we have already proved that for time-like circular geodesics the angular velocity is given by from equation (71)

$$\Omega_0 = \frac{\sqrt{(Mr_0 - Q^2)}}{r_0^2} \quad (172)$$

Again we obtained for circular null geodesics  $\Omega_c = \frac{1}{D_c}$ , so we can deduce similar expression for it is given by

$$\Omega_c = \frac{\sqrt{(Mr_c - Q^2)}}{r_c^2} . \quad (173)$$

Resultantly we obtain the ratio of angular frequency for time-like circular geodesics to the angular frequency for null circular geodesics is

$$\frac{\Omega_0}{\Omega_c} = \left( \frac{\sqrt{Mr_0 - Q^2}}{\sqrt{Mr_c - Q^2}} \right) \left( \frac{r_c^2}{r_0^2} \right) . \quad (174)$$

For  $r_0 = r_c$ ,  $\Omega_0 = \Omega_c$ , i.e., when the radius of time-like circular geodesics is equal to the radius of null circular geodesics, the angular frequency corresponds to that geodesic are equal, which demands that the intriguing physical phenomena could occur in the curved space-time, for example, possibility of exciting Quasi Normal Modes (QNM) by orbiting particles, possibly leading to instabilities of the curved space-time[7].

For  $r_0 > r_c$ , we shall prove that for Schwarzschild black-hole, Reissner Nordström black-hole the null circular geodesics have the largest angular frequency as measured by asymptotic observers than the time-like circular geodesics. We therefore conclude that null circular geodesics provide the fastest way to circle black holes[8].

Now the ratio of time period for time-like circular geodesics to the time period for null circular geodesics is given by

$$\frac{T_0}{T_c} = \left( \frac{\sqrt{Mr_c - Q^2}}{\sqrt{Mr_0 - Q^2}} \right) \left( \frac{r_0^2}{r_c^2} \right) . \quad (175)$$

This ratio is valid for  $r_0 \neq r_c$ . For  $r_0 = r_c$ ,  $T_0 = T_c$ , i.e. time period of both geodesics are equal, which possibly leading to the excitations of Quasi Normal Modes. For  $r_0 > r_c$ ,  $T_0 > T_c$ , which implies that the orbital period for time-like circular geodesics is greater than the orbital period for null circular geodesics. For  $r_0 = r_{ISCO}$  and  $r_c = r_{photon}$ ,

therefore the ratio of time period for ISCO ( $r = r_{ISCO}$ ) to the time period for photon-sphere ( $r_c = r_{photon}$ ) for Reissner Nordström black-hole is given by

$$\frac{T_{ISCO}}{T_{photon}} = \left( \frac{\sqrt{Mr_c - Q^2}}{\sqrt{Mr_0 - Q^2}} \right) \left( \frac{r_0^2}{r_c^2} \right). \quad (176)$$

This implies that  $T_{ISCO} > T_{photon}$ , therefore we conclude that ISCO provide the *slowest way* to circle the Reissner Nordström black-hole.

## 9.1 Schwarzschild Black hole

For Schwarzschild black hole,  $a = 0$ ,  $Q = 0$  we get the ratio of angular frequency for time-like circular geodesics to the angular frequency for null circular geodesics is

$$\frac{\Omega_0}{\Omega_c} = \frac{r_c \sqrt{r_c}}{r_0 \sqrt{r_0}}. \quad (177)$$

which is proportional to the radial coordinates  $r$ . Since for Schwarzschild black hole, the ISCO occurs at  $r_{ISCO} \geq 6M$  and photon sphere occurs at  $r_c = 3M$ , therefore the ratio of angular frequency for ISCO ( $r_0 = r_{ISCO} = 6M$ ) to the angular frequency for photon-sphere ( $r_c = r_{photon} = 3M$ ) is given by

$$\frac{\Omega_{ISCO}}{\Omega_{photon}} = \frac{r_c \sqrt{r_c}}{r_0 \sqrt{r_0}} = \frac{1}{2\sqrt{2}}. \quad (178)$$

Which suggests that,  $\Omega_{ISCO} < \Omega_{photon}$ , i.e ISCO is characterized by the *smallest* angular frequency as measured by the asymptotic observers than the null circular geodesics. Correspondingly the ratio of time period for time-like circular geodesics to the time period for null circular geodesics is given by

$$\frac{T_0}{T_c} = \frac{r_0 \sqrt{r_0}}{r_c \sqrt{r_c}}. \quad (179)$$

Since for Schwarzschild black hole, the ISCO occurs at  $r_{ISCO} \geq 6M$  and photon sphere occurs at  $r_c = 3M$ , therefore the ratio of time period for ISCO ( $r_0 = r_{ISCO} = 6M$ ) to the time period for photon-sphere ( $r_c = r_{photon} = 3M$ ) is given by

$$\frac{T_{ISCO}}{T_{photon}} = \frac{r_0 \sqrt{r_0}}{r_c \sqrt{r_c}} = 2\sqrt{2}. \quad (180)$$

This implies that  $T_{ISCO} > T_{photon}$ , i.e the orbital period for time-like circular geodesics is greater than the orbital period for null circular geodesics. From this we conclude that innermost stable circular orbit provide the *slowest way* to circle the black hole.

## 10 Discussion

In this article, we have used Lyapunov Exponent and Kolmogorov-Sinai entropy to give a full description of time-like circular geodesics and null circular geodesics in Reissner Nordström black hole space-time. We then explicitly derived them in terms of the equation of the radius of ISCO. We then further showed that the Lyapunov Exponent can be used to determine the stability and instability of equatorial circular geodesics, both time-like and null case for Reissner Nordström black hole space-time. Analogously, we also compute the Kolmogorov-Sinai entropy to measure the disorder-ness of a chaotic orbit as it evolves. Using them, we found that the ISCO occurs at  $r_{ISCO} = 4M$  for extremal Reissner Nordström black-hole and  $r_{ISCO} = 6M$  for Schwarzschild black-hole. We also made a link between Lyapunov Exponent and Geodesic Deviation equation. The other point we have studied that for circular geodesics around the central black-hole, ISCO is characterized by the smallest angular frequency as measured by the asymptotic observers-no other circular geodesics can have a smallest angular frequency. Thus such types of space-times always have  $\Omega_{particle} < \Omega_{photon}$  for all time-like circular geodesics. Alternatively it was shown that the orbital period for time-like circular geodesics particularly, ISCO is characterized by the greatest orbital period than the null circular geodesics. Therefore, we conclude that Innermost Stable Circular Orbit(ISCO) provide the *slowest way* to circle the black hole. The aim in future to find the ISCO for extremal Kerr and Kerr-Newman by using the Lyapunov Exponent and Kolmogorov-Sinai entropy.

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